PRINCIPLES OF ANALYSIS
LECTURE 6 - CARDINALITY

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1. Motivation

We have seen that there are infinitely many rational numbers, and infinitely many irrational numbers. So the question arises as to whether or not there are as many rational numbers as there are real numbers: there are infinitely many of both. We know that the rational numbers embed into the real numbers, but does there exist an injective function in the other direction?

We begin by demonstrating that there is more than one type of infinite set in this regard.

Proposition 1. *Let X be a set. Then there does not exist a surjective function $X \to \mathcal{P}(X)$.*

Proof. Let $f : X \to \mathcal{P}(X)$; we wish to show that f is not surjective. Set

$$Y = \{x \in X \mid x \notin f(x)\}.$$

Suppose, by way of contradiction, that $f(x) = Y$ for some $x \in X$. Is $x \in Y$? If it is, then $x \in f(x)$, so by definition of Y, $x \notin Y$. On the other hand, if it is not, then $x \notin f(x)$, so $x \in Y$. Either case is an immediate contradiction. Thus there is no such x satisfying $f(x) = Y$, and Y is not in the image of f. Therefore f is not surjective. □

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2. Cardinal Numbers

Let U be a set; we refer to U as a *universal set*, and assume that U contains $\mathbb{R}$.

Let A and B be sets. We say that A and B have the same *cardinality* if there exists a bijective function between them. If A and B have the same cardinality, we write $A \sim B$. Then $\sim$ is a relation on $\mathcal{P}(U)$.

Proposition 2. *The relation $\sim$ is an equivalence relation on $\mathcal{P}(U)$.*

We shall call the equivalence classes of the relation the *cardinal numbers in U*. Let $\gimel$ denote the set of cardinal numbers in U. If $A \subset U$, the equivalence class to which it belongs is denoted $|A|$, and is called the *cardinality* of A.

Define a relation $\leq$ on $\gimel$ by

$$|A| \leq |B| \Leftrightarrow \exists \text{ injective } f : A \to B;$$

where $A, B \subset U$ are representatives of the cardinal numbers $|A|$ and $|B|$ respectively.

Proposition 3. *The relation $\leq$ on $\gimel$ is well defined.*

That is, let $A_1, A_2, B_1, B_2 \subset U$ such that $A_1 \sim A_2$ and $B_1 \sim B_2$, and such that $|A_1| \leq |B_1|$. Show that $|A_2| \leq |B_2|$.

3. Schroeder-Bernstein Theorem

Lemma 1 (Banach's Lemma)**.** *Let X and Y be sets. and let $f : X \to Y$ and $g : Y \to X$ be injective functions. There exist subsets $A \subset X$ and $B \subset Y$ such that $f(A) = B$ and $g(Y \setminus B) = X \setminus A$.*

Proof. Fix the following objects:

- Let X and Y be sets.
- Let $f : X \to Y$ and $g : Y \to X$ be injective functions.
- Let $h = g \circ f$.
- Let $C_0 = X \setminus g(Y)$.
- Let $C_n = h(C_{n-1})$, for each $n \in \mathbb{N}$.
- Let $A = \cup_{n=0}^{\infty} C_n$.
- Let $B = f(A)$.

It suffices to show that $g(Y \setminus B) = X \setminus A$.

Claim 1: $h(A) \subset A$.
Let $a_0 \in h(A)$. Then $a_0 = h(a_1)$ for some $a_1 \in A$. By definition of A, $a_1 \in C_n$ for some $n \in \mathbb{N}$. Then $a_0 \in C_{n+1}$. Thus $a_0 \in A$.

Claim 2: $g(Y \setminus B) \subset X \setminus A$.
We want to select an arbitrary $y_0 \in Y \setminus B$ and show that g sends it into $X \setminus A$. Let $x_0 \in g(Y \setminus B)$. Then there exists $y_0 \in Y \setminus B$ such that $g(y_0) = x_0$. Suppose bwoc that $x_0 \in A$. Since $x_0 \in g(Y)$, $x_0 \notin C_0$, so $x_0 \in C_n$ for some $n > 0$. Since $C_n = h(C_{n-1})$, there exists $x_1 \in C_{n-1}$ such that $h(x_1) = x_0$. So $g(f(x_1)) = x_0$. Since g is injective, $f(x_1) = y_0$. But $x_1 \in A$, so $y_0 \in B$. This is a contradiction. Thus $x_0 \notin A$, so $x_0 \in X \setminus A$. Since x_0 was chosen arbitrarily, $g(Y \setminus B) \subset X \setminus A$.

Claim 3: $g(Y \setminus B) \supset X \setminus A$.
We want to select an arbitrary $x_0 \in X \setminus A$ and find $y_0 \in Y \setminus B$ which g sends to it. Let $x_0 \in X \setminus A$. Since $C_0 \subset A$, then $x_0 \in X \setminus C_0$. That is, $x_0 \in g(Y)$, so there exists $y_0 \in Y$ such that $g(y_0) = x_0$. Suppose bwoc that $y_0 \in B$. Then there exists $x_1 \in A$ such that $f(x_1) = y_0$. Thus $h(x_1) = x_0$, so $x_0 \in h(A)$. Since $h(A) \subset A$, $x_0 \in A$, which is a contradiction. Thus $y_0 \notin B$, so $x_0 \in g(Y \setminus B)$. Since x_0 was chosen arbitrarily, $X \setminus A \subset g(Y \setminus B)$. □

Theorem 1 (The Schroeder-Bernstein Theorem)**.** *Let X and Y be sets. If there exist injective functions $f : X \to Y$ and $g : Y \to X$, then there exists a bijective function $h : X \to Y$.*

Proof. Let A and B be sets as specified by the lemma. Let $V = X \setminus A$ and $W = Y \setminus B$. Then $f \upharpoonright_A : A \to B$ is bijective, and $g \upharpoonright_W : W \to V$ is bijective. Let $r = (g \upharpoonright_W)^{-1}$. Then $r : V \to W$ is bijective. Thus define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ r(x) & \text{if } x \in V. \end{cases}$$

□

4. Axiom of Choice

Assume the following version of a famous axiom from set theory.

Axiom 1. Axiom of Choice
Let A and B be sets.

(a) There exists either a surjective function $A \to B$ or a surjective function $B \to A$.

(b) There exists an injective function $f : A \to B$ if and only if there exists a surjective function $g : B \to A$.

Corollary 1. *Let X and Y be sets. If there exist surjective functions $f : X \to Y$ and $g : Y \to X$, then there exists an bijective function $h : X \to Y$.*

Proof. This follows immediately by combining the Schroder-Bernstein Theorem with the Axiom of Choice. □

Corollary 2. *Let X and Y be sets. The following conditions are equivalent:*

- $|X| = |Y|$*;*
- $\exists$ *a bijective function* $X \to Y$*;*
- $\exists$ *injective functions* $X \to Y$ *and* $Y \to X$*;*
- $\exists$ *surjective functions* $X \to Y$ *and* $Y \to X$*.*

Proposition 4. *Show that* $(\beth, \leq)$ *is an ordered set.*

Proof. To show this, one must show that $\leq$ is a total order relation on $\mathcal{P}(U)$. The proof of symmetry uses the Schroder Bernstein Theorem, and the proof of definiteness requires the Axiom of Choice. □

The total order relation $\leq$ on $\beth$ naturally leads to the following definitions for derived relations on $\beth$:

- $|A| \geq |B| \Leftrightarrow |B| \leq |A|$;
- $|A| < |B| \Leftrightarrow \neg(|A| \geq |B|)$;
- $|A| > |B| \Leftrightarrow \neg(|A| \leq |B|)$.

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